ALPCAHUS: Heteroscedastic Subspace Clustering

Javier Salazar Cavazos

Joint collaboration with
Laura Balzano & Jeff Fessler

University of Michigan

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Image Credit: ???

Preface

1. Originally, presentation about convergence guarantees (nonconvex)

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2. Gave a presentation a year ago about Sparse Subspace Clustering
3. Goal: Heteroscedastic version of SSC
4. Result thus far = pain :(
5. Instead, worked on heteroscedastic PCA (quals presentation)
6. This presentation covers ALPCAH $\rightarrow$ Union of Subspace model

Subspaces

For a vector space $\mathcal{V}$ defined on a field $\mathbb{F}$, a nonempty set $\mathcal{S} \subseteq \mathcal{V}$ is called a *linear* subspace iff

- S is closed under vector addition (i.e. for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{S} \implies \boldsymbol{u} + \boldsymbol{v} \in \mathcal{S}$)
- S is closed under scalar multiplication (i.e. $\boldsymbol{v} \in \mathcal{S} \implies \alpha \boldsymbol{v} \in \mathcal{S}$)

Key Points:

- Every *linear* subspace includes $\mathbf{0}$
- A shifted *linear* subspace is called an affine subspace
- Left singular matrix contains subspace basis

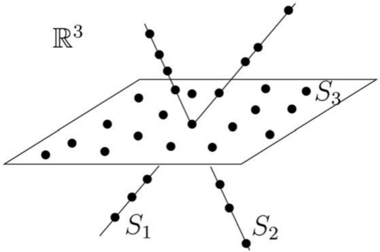
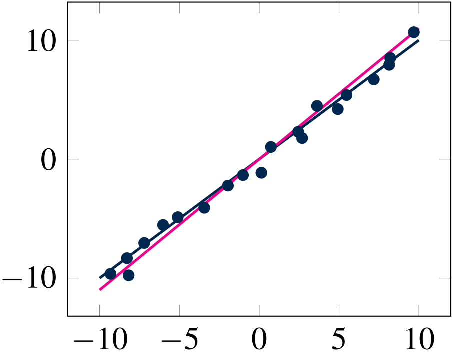


Image Credit: René Vidal @ Johns Hopkins University/University of Pennsylvania

Heteroscedasticity

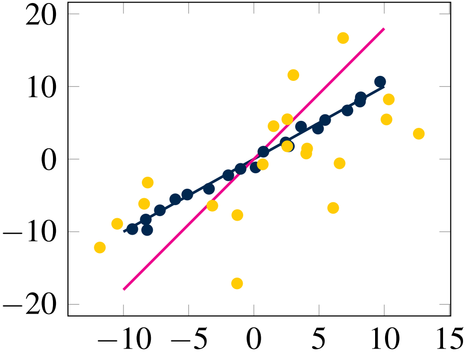
Homoscedastic Data

$$\boldsymbol{y}_i = \boldsymbol{x}_i + \boldsymbol{\epsilon} \quad \text{s.t.} \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \nu I)$$



Heteroscedastic Data

$$\boldsymbol{y}_i = \boldsymbol{x}_i + \boldsymbol{\epsilon}_i \quad \text{s.t.} \quad \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \nu_i I)$$



$\boldsymbol{x}_i = U\boldsymbol{z}_i$ where $\hat{U}$ = estimated subspace basis, $\boldsymbol{z}_i$ = basis coordinates

ALPCAH [1] (Algorithm Low-rank PCA Hetero. data)

Key Idea

ALPCAH = Robust PCA (RPCA) + Heteroscedastic Noise

Let $Y = [\, \boldsymbol{y}_1, ..., \boldsymbol{y}_n \,] \in \mathbb{R}^{D \times N}$ and $\Pi = \text{diagm}(\nu_1, \ldots, \nu_n) \in \mathbb{R}^{N \times N}$

Decompose Y such that $\underbrace{Y}_{\text{data matrix}} = \underbrace{X}_{\text{low rank data}} + \underbrace{Z}_{\text{noise matrix}}$

Then the optimization problem we posed is the following:

$$\arg\min_{X,Z,\Pi} \underbrace{\lambda f_k(X)}_{\text{low rank}} + \underbrace{\frac{1}{2} \|Z\Pi^{-1/2}\|_F^2}_{\text{weighted noise}} + \underbrace{\frac{D}{2} \log \overbrace{|\Pi|}^{\text{det.}}}_{\text{unk. variance}} \quad \text{s.t.} \quad Y = X + Z$$

ALPCAH [1] (Algorithm Low-rank PCA Hetero. data)

Let $Y = [\, \boldsymbol{y}_1, ..., \boldsymbol{y}_n \,] \in \mathbb{R}^{D \times N}$ and $\Pi = \text{diagm}(\nu_1, \ldots, \nu_n) \in \mathbb{R}^{N \times N}$

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$$f_k(X) \triangleq \sum_{i=k+1}^{\min(M,N)} \sigma_i(X) = \sum_{i=1}^{\min(M,N)} \sigma_i(X) - \sum_{i=1}^{k} \sigma_i(X) = \|X\|_* - \|X\|_{\text{Ky-Fan}(k)}$$

Note!

$k = 0 \implies f_k(X) = \|X\|_*$ (convex assuming known variance)

$k > 0, \lambda \to \infty \implies \hat{X} \implies SVP(X, \alpha) = U_k \Sigma_k V_k^T$ (nonconvex)

Matrix Factorized ALPCAH

Key Idea

FAST ALPCAH = ALPCAH + Matrix Factorization

Let $Y = [\,\boldsymbol{y}_1, ..., \boldsymbol{y}_n\,] \in \mathbb{R}^{D \times N}$ and $\Pi = \text{Diagonal}(\nu_1, \ldots, \nu_n) \in \mathbb{R}^{N \times N}$

Decompose X such that $X = LR'$ where $L, R \in \mathbb{R}^{D \times k}$

Then the optimization problem we posed is the following:

$$\underset{L,R,\Pi}{\arg\min} \underbrace{\frac{1}{2}\|(Y - LR')\Pi^{-1/2}\|_F^2}_{\text{new term}} + \frac{D}{2}\log|\Pi|$$

note: not the focus for today, but performs relatively well quality-wise

Subspace Clustering

Motivation

Face Clustering [2]

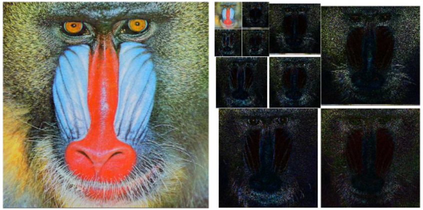


Image Compression [3]

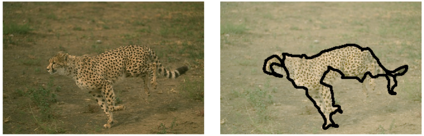
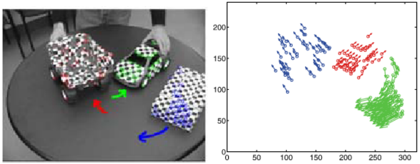


Image Segmentation [4]



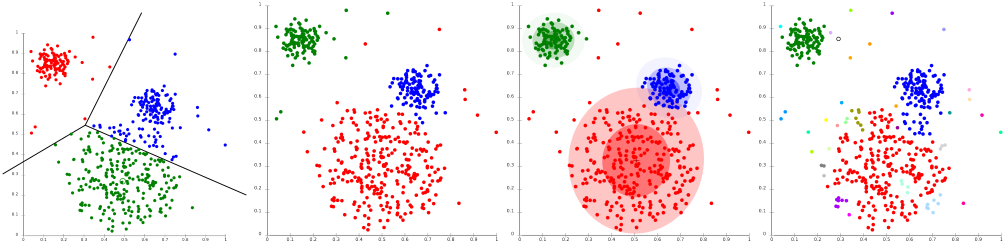
Motion Estimation [5]

Image Credit: Respective Papers

Clustering

An unsupervised machine learning method to identify and group "similar" unlabeled data points in order to find structure or patterns

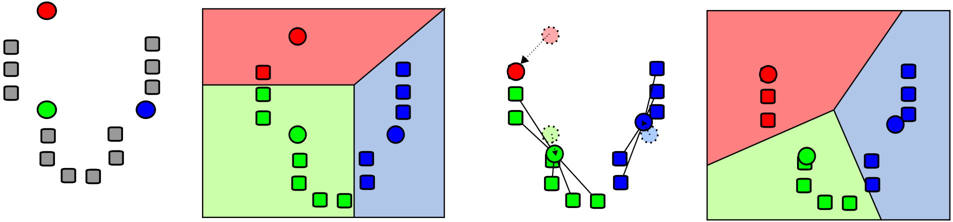
By "similar", this can mean different things:



Centroid-based Density-based Model-based Hierarchical-based

Image Credit: Chire @ Wikipedia, CC BY-SA 3.0

K-Means Clustering



1. Initialize centroids
2. Generate clusters
3. Update centroids
4. Repeat steps 2 & 3

Note!

NP-hard problem $\Longrightarrow$ initialization is extremely important!

Subspaces + Clustering

Points:

$Y = [\, y_1, \ldots, y_N \,]$ for $\{\, y_i \in \mathbb{R}^D \,\}_{i=1}^N$ drawn from $\{U_i\}_{i=1}^K$

Subspaces:

$\mathcal{U} = \{\, U_i \in \mathbb{R}^{D \times d_i} \} \ \text{s.t.}\ \exists \alpha \in \{1, \ldots, K\} \rightarrow y_i = U_\alpha z_i + \epsilon_i$

Clusters:

$\mathcal{C} = \{\, c_i \ : \ c_i \subset \{1, \ldots, N\}\} \ \text{s.t.}\ Y_{(c_i)} \in \mathbb{R}^{D \times |c_i|} \subset Y \in \mathbb{R}^{D \times N}$

Goal:

Find subspace bases $\mathcal{U}$ and clusters $\mathcal{C}$

Literature Review [6]

Algebraic Methods (e.g. Generalized PCA)

- Geometric and algebraic interpretations

Iterative Methods (e.g. K-Subspaces)

- Alternate between segmentation and subspaces

Statistical Methods (e.g. Mixture of Probabilistic PCA)

- Model assumptions combined with prior beliefs

Affinity-based Methods (e.g. Sparse Subspace Clustering)

- Measures “similarity” between points

ALPCAHUS (updated)

K-Subspaces (KSS)

KSS = K-means + Subspaces

KSS cost function [7]:

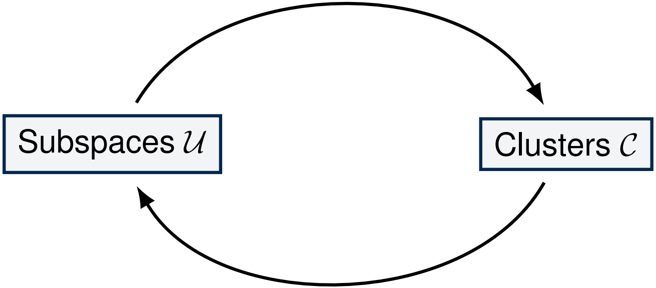
$$\min_{\mathcal{C},\mathcal{U}} \sum_{k=1}^{K} \sum_{i:\ i \in c_k} \|y_i - U_k U_k^T y_i\|_2^2$$

ALPCAHUS cost function:

$$\min_{\mathcal{C},\Pi,\mathcal{L},\mathcal{R}} \sum_{k=1}^{K} \frac{1}{2} \|[Y_{(c_k)} - \mu_{(c_k)} 1' - L_{(c_k)} R'_{(c_k)}] \Pi_{(c_k)}^{-1/2}\|_F^2 + \frac{D}{2} \log |\Pi_{(c_k)}|$$

Intuition

$$\min_i \underbrace{\|y_p - \overbrace{U_i U_i'}^{\text{subspace projector}} y_p\|}_{\text{residual}} \; \forall y_p \implies p \in c_i$$

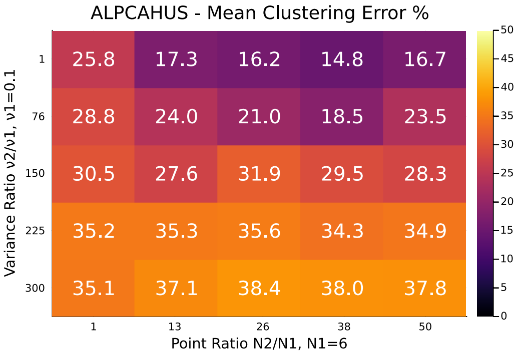


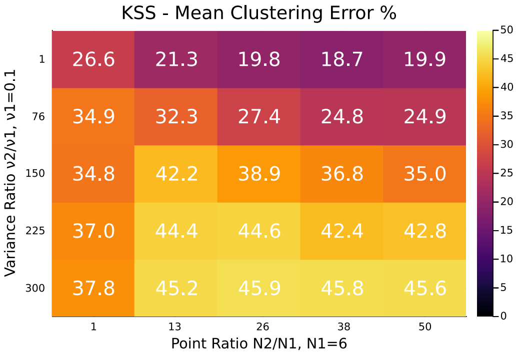
$$\text{ALPCAH}(Y_{(c_i)}) \; \forall c_i \implies U_i$$

Experimental Setup

1. Gaussian random matrix $\rightarrow$ SVD $\rightarrow U_k \;\; \forall k \in \{1, 2\} \;\; d_k = 3$
2. Uniform random vector $\rightarrow z_i \sim U[-10, 10] \rightarrow x_i = U_k z_i$
3. Gaussian noise $\rightarrow \epsilon_i \sim \mathcal{N}(0, \nu_i I) \rightarrow y_i = x_i + \epsilon_i$
4. Each subspace composed of data with noise variances $\nu_1 = 0.1, \nu_2$
5. Each subspace contains good data and bad data $N1 = 6, N2$
6. Mean clustering error will be measured (misclassification rate)

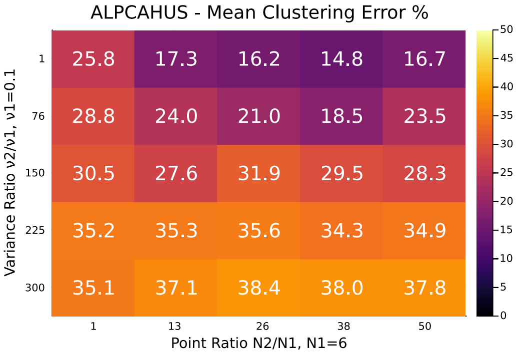
ALPCAHUS vs. KSS - Result I

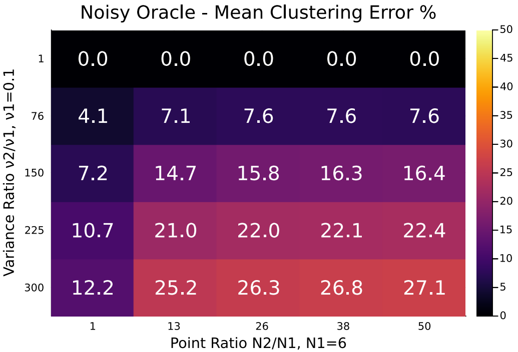




always performs better, even in homoscedastic setting?

ALPCAHUS vs. KSS - Result II





Is it possible to get closer?

question: what is a disadvantage of KSS?

Ensemble KSS (EKSS) [7]

Key Idea

KSS very sensitive to initialization $\therefore$ leverage info from many trials!

For each trial $b \in \{1, \ldots, B\}$, collect all clustering results $C = [c_1, \ldots, c_B]$

Form co-association matrix (affinity matrix)
$A_{ij} \leftarrow \frac{1}{B}|\{b \ : \ x_i, x_j \text{ are co-clustered in } \mathcal{C}^{(b)}\}|$

essentially, points classified similarly have a high affinity

Affinity Matrix

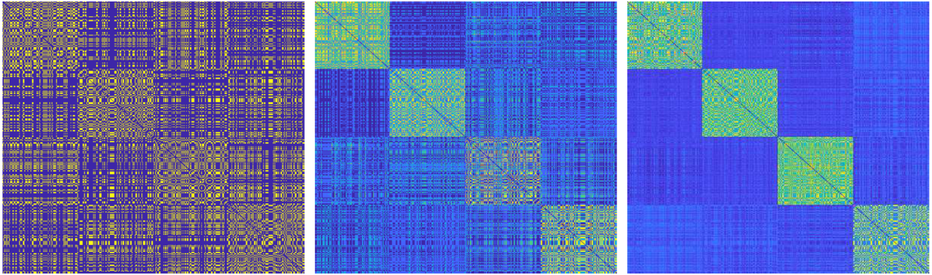


Fig. 1. Co-association matrix of EKSS for $B = 1, 5, 50$ base clusterings. Data generation parameters are $D = 100$, $d = 3$, $K = 4$, $N = 400$, and the data is noise-free; the algorithm uses $\bar{K} = 4$ candidate subspaces of dimension $\bar{d} = 3$ and no thresholding. Resulting clustering errors are 61%, 25%, and 0%.

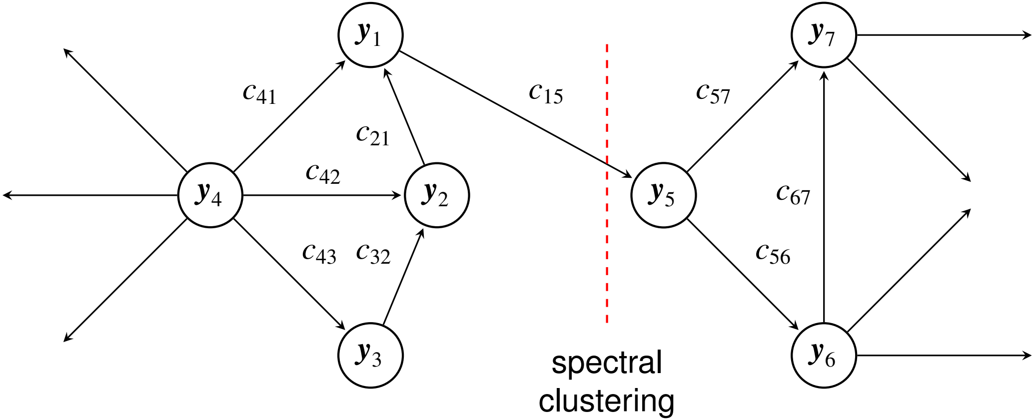
note: structure is nice only because data is ordered!

question: how do we get the clusters from this matrix?

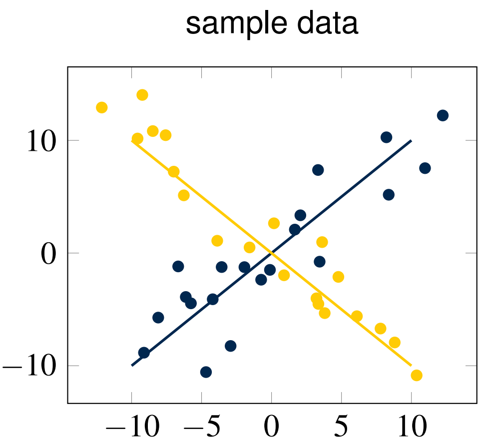
Image Credit: Authors from [7]

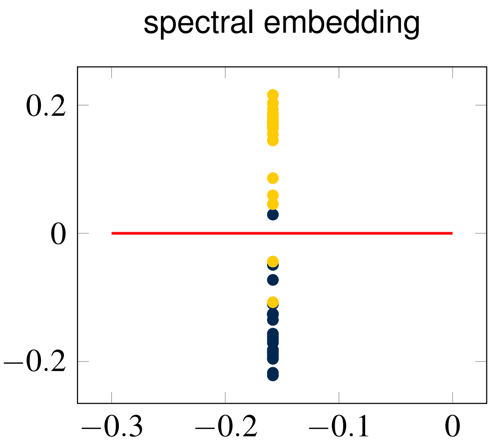
Spectral Clustering [8]

Data points $\Longrightarrow$ nodes/vertices
coefficients $\Longrightarrow$ edges



Toy Example



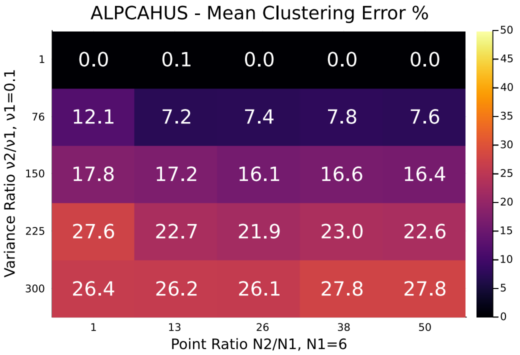


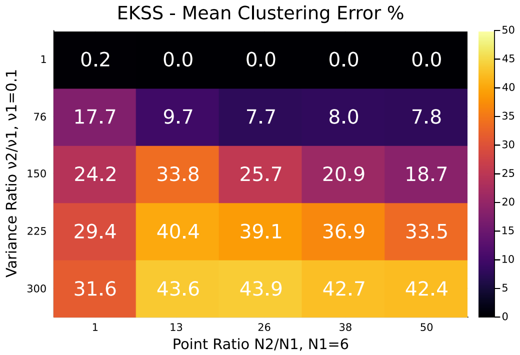
$$\text{EKSS}(Y) \to A \to D_{ii} = \sum_j A_{ij} \to L = I - D^{-1}A \to \text{EVD}(L) \to V'_K \to \text{kmeans}(V'_K)$$

Short Summary

1. Run EKSS and form affinity matrix A
2. Threshold matrix to reduce false connections (not discussed)
3. Perform spectral clustering on A to get final clusters
4. Perform PCA on each cluster to get subspaces

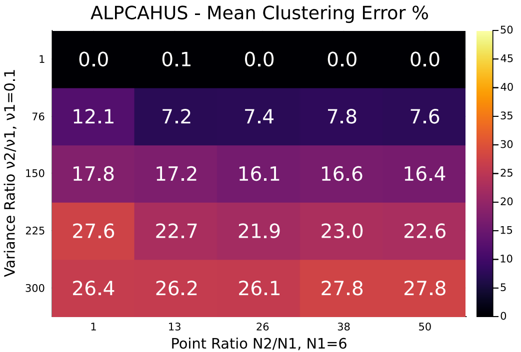
ALPCAHUS vs. EKSS - Result I

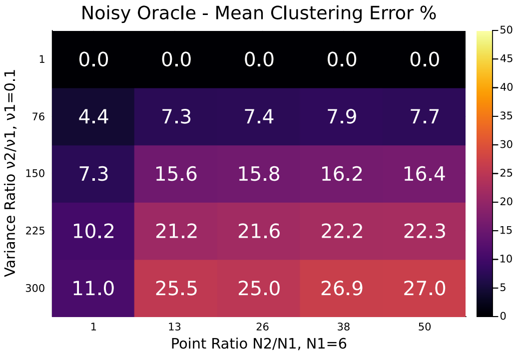




we are able to better learn the subspaces!

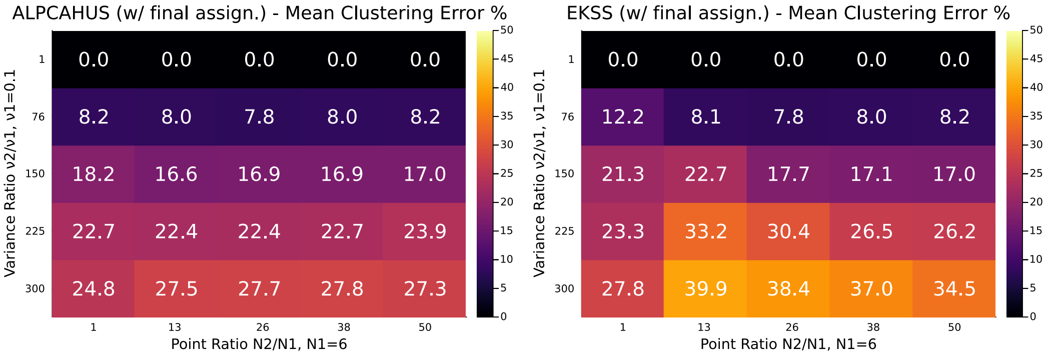
ALPCAHUS vs. EKSS - Result II





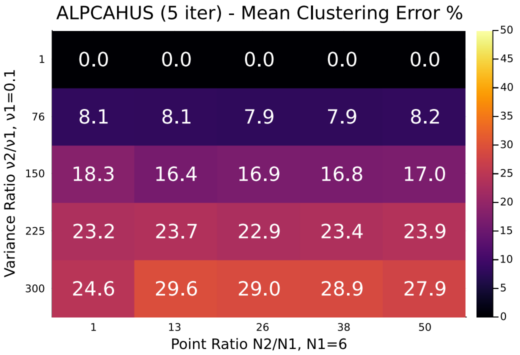
now we are much closer!

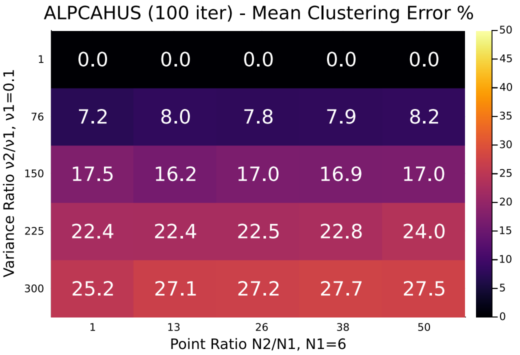
ALPCAH After EKSS?



EKSS final cluster is still too noisy!

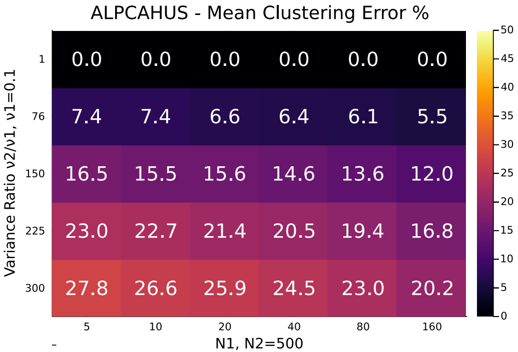
ALPCAH Iterations

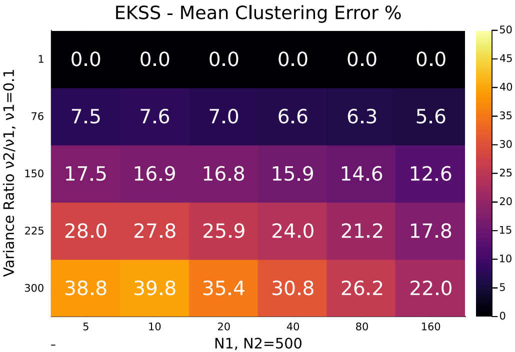




not many iterations needed for ALPCAH!

Good Data Experiment





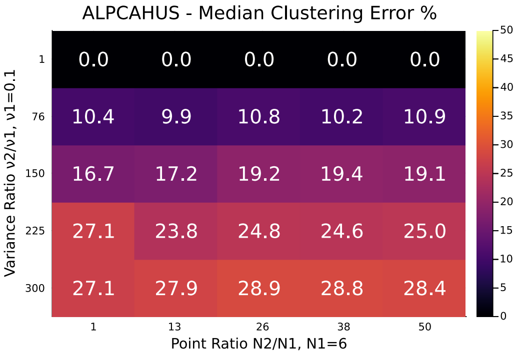
works for a “large” range of good data!

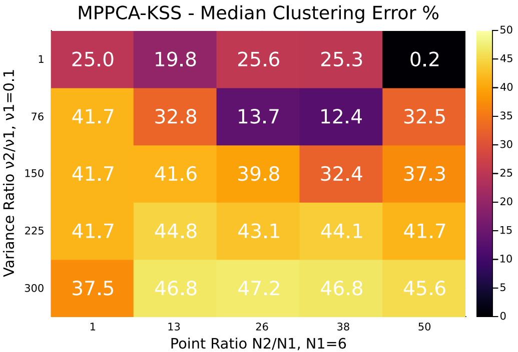
Conclusion

1. ALPCAHUS might be beneficial in homo. setting (not discussed)
2. Heteroscedasticity is problematic for various research areas like SC
3. Both the clusterings and subspace estimations are thrown off
4. Without knowing noise variances, we can account for this kind of data
5. ALPCAH + KSS can improve the clustering/subspaces estimates
6. No need to solve ALPCAH exactly, few iterations are enough

Future Work

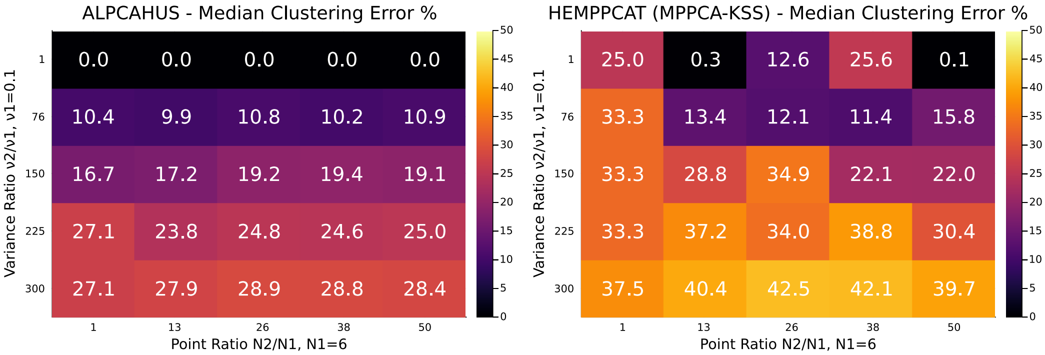
1. This is very early work!
2. Real data examples? COIL-100? (homoscedastic but that might be ok)
3. Laura gave me an idea for estimating subspace dimensions
4. More comparisons with other algorithms like SSC,TSC, . . .
5. Thanks for listening! Suggestions? Comments?





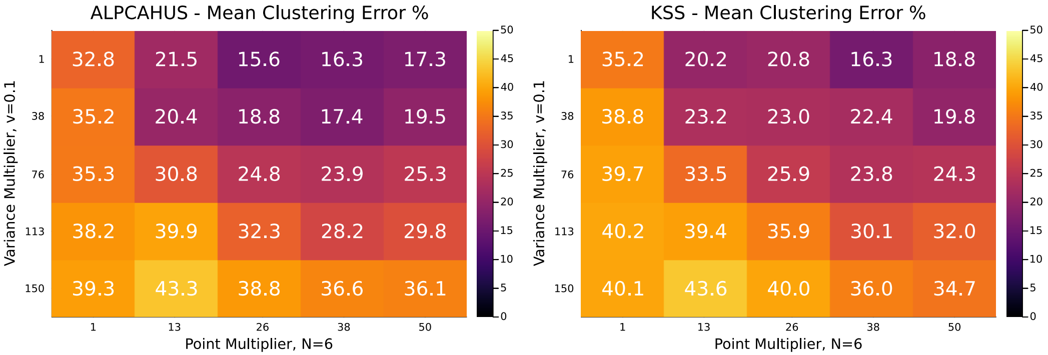
MPPCA assumes each mixture has the same variance

HEMPPCAT [10]



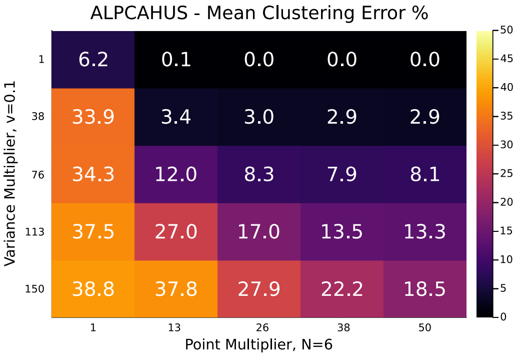
ALPCAHUS outperforms heteroscedastic MPPCA

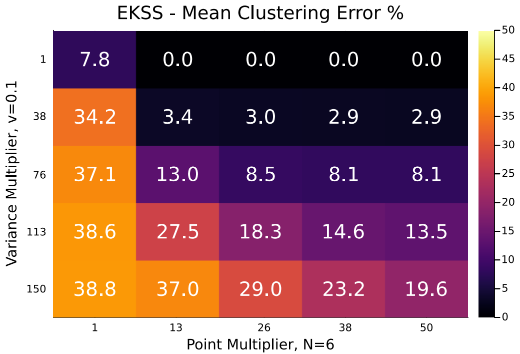
Homoscedastic Setting (KSS)



ALPCAHUS can be beneficial in homoscedastic setting

Homoscedastic Setting (Ensemble KSS)





No longer true for the ensemble methods

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